QUANTUM AND CLASSICAL DERIVATIONS OF BLACK HOLE ENTROPY AND PHYSICAL MICROSTATES

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Abstract

Classical and quantum mechanical derivations of black hole entropy have been studied in detail in the frame work of general relativity ,thermodynamics and quantum mechanics. The black hole entropy and area have been derived from classical thermodynamics. The relationship between the classical and quantum formulas can be shown to be similar to that of black body radiation. Microstates of the black hole entropy and its physical implications have been investigated. Some applications of black hole entropy to astrophysics have also been presented.

Keywords: Black hole entropy.

Introduction

When the core runs out of hydrogen fuel, it will contract under the weight of gravity. However, some hydrogen fusion will occur in the upper layers. As the core contracts, it heats up. This heats the upper layers, causing them to expand. As the outer layers expand, the radius of the star will increase and it will become a red giant. The radius of the red giant sun will be just beyond Earth's orbit. At some point after this, the core will become hot enough to cause the helium to fuse into carbon. When the helium fuel runs out, the core will expand and cool. The upper layers will expand and eject material that will collect around the dying star to form a planetary nebula. Finally, the core will cool into a white dwarf and then eventually into a black dwarf. This entire process will take a few billion years.

When the core runs out of hydrogen, these stars fuse helium into carbon just like the sun. However, after the helium is gone, their mass is enough to fuse carbon into heavier elements such as oxygen, neon, silicon, magnesium, sulfur and iron. Once the core has turned to iron, it can burn no longer. The star collapses by its own gravity and the iron core heats up. The core becomes so tightly packed that protons and electrons merge to form neutrons.

In less than a second, the iron core, which is about the size of Earth, shrinks to a neutron core with a radius of about 6 miles (10 kilometers). The outer layers of the star fall inward on the neutron core, thereby crushing it further. The core heats to billions of degrees and explodes (supernova), thereby releasing large amounts of energy and material into space. The shock wave from the supernova can initiate star formation in other interstellar clouds. The remains of the core can form a neutron star or a black hole depending upon the mass of the original star.

In the ordinary evolution of very massive stars, black holes can be formed. A star is essentially a gigantic nuclear reactor converting hydrogen to helium in a process called nuclear fusion. Think to the star as millions of hydrogen bombs going off at the same time, thereby producing enormous quantities of energy and enormous forces outward from the star. There is an equilibrium between the gravitational forces inward and the forces outward caused by the exploding gases. Eventually, when all the nuclear fuel is used, there is no longer an equilibrium

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condition. The gravitational force causes the gas to become very compact. If the star is large enough, it is compressed below its Schwarzschild radius and a black hole is formed. For an evolving star to condense into a black hole it must be approximately 25 times the mass of the sun. When the star condenses to a black hole it does not stop at the event horizon but continues to reduce in size until it becomes a singularity, a point mass. That is, the entire mass of the star has condensed to the size of a point.

A star that was initially more massive than about 20reaches the end of its life and collapses, it may create a compact star whose properties differ dramatically from those of white dwarfs or neutron stars. Its greater mass can compress its core so much that pressure is unable to support it, and it totally collapses to form what astronomers call a black hole.(Bekenstein, J.D., 1973)

A black hole is at once the most simple and the most complex object. It is the most simple in that it is completely specified by its mass, spin, and charge. This remarkable fact is a consequence of the so called 'No Hair Theorem'. It is the most complex in that it possesses a huge entropy. The entropy of a solar mass black hole is enormously bigger than the thermal entropy of the star that might have collapsed to form it. Entropy gives an account of the number of microscopic states of a system.

Black holes have very strange properties, and to understand them one needs to review the concept of escape velocity. For a body of mass M and at a radius R from the center of that object , the escape velocityv_{esc} , for an object to travel away from the body is

$$v_{esc}=\sqrt{\frac{2GM}{R}}$$

Where, G = the gravitational constant (6.67×10^{-11}Nm^2/ kg^2)

v_{esc}= the escape velocity (meter per seconds)

R = radius (meters)

M = mass (kilograms)

One can see from the formula that the escape velocity for a body of a given mass will be larger at a smaller radius. When the escape velocity is greater than the speed of light, such an object would become a black hole. (Raine D & Thomas E., 2005)

History of Black Hole Entropy

In 1972, Bekenstein was the first to suggest that black holes should have a well-defined entropy. He wrote that a black hole's entropy was proportional to the area of its (the black hole's) event horizon. Bekenstein also formulated the generalized second law of thermodynamics, black hole thermodynamics, for systems including black holes. Based on his black-hole thermodynamics work, Bekenstein also demonstrated the Bekenstein bound: there is a maximum to the amount of information that can potentially be stored in a given finite region of space which has a finite amount of energy. In 1982, Bekenstein developed a rigorous framework to generalize the laws of electromagnetism to handle inconstant physical constants. His framework replaces the fine-structure constant by a scalar field. However, this framework for changing constants did not incorporate gravity. In 2004, Bekenstein boosted Mordehai Milgrom's theory of Modified Newtonian Dynamics (MOND) by developing a relativistic version.

Meaning of Black Hole Entropy

Black holes are really thermodynamic systems with an actual temperature and entropy. The entropy should be the logarithm of the number of independent states of the black hole.

The Fact that the black hole entropy is even finite is already puzzling. A box of radiation at fixed energy and volume has a finite entropy because the box imposes a long wavelength cut off and the total energy imposes a short wavelength cut off. The Hilbert space describing the radiation field inside the box at fixed energy is thus finite dimensional, and the microcanonical entropy is just the logarithm of its dimension. A black hole in a box at fixed energy would also have a short wavelength cutoff (at the box) but, as emphasized by Hooft, according to standard quantum field theory it has no long wavelength cutoff (at the box). The reason is that the horizon is an infinite redshift surface. The wave vector of any outgoing mode diverges at the horizon, and is red shifted down to a finite value at the box. The entropy of each radiation field around a black hole is therefore infinite due to a divergence in the mode density at the horizon, so it seems the black hole entropy must also diverge. This divergence is equivalent to a divergence in the renormalization of Newton's constant, or rather in $\frac{1}{G}$ .Thus one point of view is that it should be absorbed by "counter terms", and only the total, renormalized entropy is relevant.(Shapiro, S.L , 1983)

Laws of Black Hole Mechanics

1. **Zeroth Law** : The temperature T of body at thermal equilibrium is constant throughout the body. Heat will flow from hot spots to the cold spots. For stationary black hole, one can show that surface gravity k is constant on the event horizon. This is obvious for spherically symmetric horizons but is true also more generally for non-spherical horizons of spinning black holes.
2. **First Law**: Energy is conserved, $dE = T\,dS + \mu\,dQ + \Omega\,dJ$, where E is the energy , Q is the charge with chemical potential μ and J is the spin with chemical potential Ω. For black hole , one has $dM = \frac{\kappa}{8\pi} dA + \mu\,dQ + \Omega\;dJ$ .For a Schwarzschild black hole one has $\mu = \Omega = 0$ because there is no charge or spin.
3. **Second Law**: The total entropy S near decreases, ΔS ≥ 0. For black holes one can prove the area theorem that the net area in any process never decreases, ΔA ≥ 0. For example, two Schwarzschild black holes with masses M_1 and M_2 can coalesce to form a bigger black hole of mass M. So the area is proportional to the square of the mass and $(M_1 + M_2)^2 \geq M_1{}^2 + M_2{}^2$.

Derivation of Black Hole Entropy

If a black hole has energy E and entropy S, then it must also have temperature T is given by

$$\frac{1}{T} = \frac{\partial S}{\partial E}$$

Where, T = temperature, E = energy, S = entropy

For a Schwarzschild black hole, the area and the entropy scales as $S \sim M^2$.

$$\frac{1}{T} = \frac{\partial S}{\partial M} \approx \frac{\partial M^2}{\partial M} \approx M \quad (1)$$

$$T = \frac{\partial M}{\partial S} = \frac{dM}{dS}$$

$$dM = TdS \quad (2)$$

Where, A_H = the area of event horizon

For Schwarzschild Black Hole's,

$$A_H = 16\pi\, G^2 M^2 \ (\because\ r_H = 2GM\ ) \quad (3)$$

Differentiating equation (3) with respect of M.

$$\frac{dA_H}{dM} = 16 \times 2\pi\, G^2 M$$

$$dA_H = 32\pi\, G^2 M dM$$

$$dM = \frac{dA_H}{32\pi\, G^2 M} \quad (4)$$

For Schwarzschild black hole,

$$\kappa = \frac{1}{4GM}$$

$$\frac{1}{M} = 4G\kappa \quad (5)$$

Substituting equation (5) in equation (6)

$$dM = \frac{dA_H}{32\pi\, G^2} \times 4G\kappa$$

$$= \frac{\kappa\, dA_H}{8\pi\, G}$$

$$= \frac{\kappa\, \hbar}{8\pi\, G\hbar} dA_H \quad (6)$$

Hawking calculation showed that the spectrum emitted by the black hole is precisely thermal with temperature,

$$T = \frac{\hbar\kappa}{2\pi},$$

Since $$\kappa = \frac{1}{4GM}, \quad T = \frac{\hbar}{8\pi\, GM} \quad (7)$$

Substituting equation (7) in equation (2), one gets

$$dS = \frac{8\ \pi\ GM}{\hbar} \times \frac{\kappa\ \hbar}{8\pi\ G\ \hbar} dA$$

$$= \frac{\kappa M}{\hbar} dA$$

$$= \frac{1}{4GM} \times \frac{M}{\hbar} dA$$

$$dS = \frac{1}{4G\hbar} dA \tag{8}$$

Taking integration to both sides of equation (8), one gets

$$\int dS = \frac{1}{4G\hbar} \int dA$$

$$S = \frac{A}{4G\hbar}$$

Since, G= $\hbar$ =1(in natural units), on now gets

$$S = \frac{A}{4} \tag{9}$$

Theoretical Validity of S=F(A)

The first law of thermodynamics applied to the system black hole surrounding may be written as

$$dM = TdS - dW \tag{10}$$

where, W= work done on the black hole

M= mass of the black hole

T= temperature

S= entropy

The work done due to changes in angular momentum and electric charge is

$$dW = \Phi\ dQ - \Omega\ dJ \tag{11}$$

where, Φ = the electric potential on the event horizon

Ω = the angular velocity on the event horizon

J = angular momentum of the black hole

Q= charge of the black hole

Substituting equation (11) in equation (10) gives,

$$TdS = dM - \Phi\ dQ - \Omega\ dJ \tag{12}$$

$$\Phi = \frac{4\pi\ r_+ Q}{A}, \Omega = \frac{4\pi\ a}{A} \tag{13}$$

where $r_+ = M + \sqrt{M^2 - Q^2 - a^2}$ and $A = 4\pi(r_+^2 + a^2), a \equiv \frac{J}{M}$

r_+ = radius of the event horizon

A = area of the event horizon

a = specific angular momentum

The first law of black hole mechanics:

$$\frac{1}{8\pi}\kappa\, dA = dM - \Phi\, dQ - \Omega\, dJ \tag{14}$$

where κ is the surface gravity given by

$$\kappa = 8\pi\left(\frac{\partial M}{\partial A}\right)_{J,Q} \tag{15}$$

Kerr-Newman solution is given by,

$$\kappa = \frac{r_+ - r_-}{\left(r_+^2 + a^2\right)} = \frac{\left(M + \sqrt{M^2 - Q^2 - a^2}\right) - \left(M - \sqrt{M^2 - Q^2 - a^2}\right)}{2\left(r_+^2 + a^2\right)}$$

$$\kappa = \frac{M + \sqrt{M^2 - Q^2 - a^2} - M + \sqrt{M^2 - Q^2 - a^2}}{2\left(r_+^2 + a^2\right)}$$

$$\kappa = \frac{2\sqrt{M^2 - Q^2 - a^2}}{2\left(r_+^2 + a^2\right)}$$

$$\kappa = \frac{4\pi\sqrt{M^2 - Q^2 - a^2}}{4\pi\left(r_+^2 + a^2\right)}$$

$$\kappa = \frac{4\pi\sqrt{M^2 - Q^2 - a^2}}{A} \tag{16}$$

According to equation (15) and (16), one has

$$\kappa = 8\pi\left(\frac{\partial M}{\partial A}\right)_{J,Q} = \frac{4\pi\sqrt{M^2 - Q^2 - a^2}}{A} \tag{17}$$

Combining equation (12) and (14) gives

$$\frac{1}{8\pi}\kappa\, dA = T\, dS\ . \tag{18}$$

Integrating both sides of equation (18), one gets

$$\frac{\kappa}{8\pi}\int dA = T\int dS$$

$$\frac{\kappa A}{8\pi} = TS \tag{19}$$

$$S = \frac{\kappa A}{8\pi T} \tag{20}$$

The black hole entropy must be a definite function of its horizon area. Therefore, it reads:

$$S = F(A) \tag{21}$$

Microscopic Derivation of the Black Hole Entropy

The Bekenstein-Hawking black hole entropy is given by

$$S = \frac{Area}{G} = \frac{\pi\sqrt{16GMl^2 + 2r_+^2}}{G} \tag{22}$$

where, M = mass ,

J =angular momentum

G = gravitational constant

r_+ =radius of the event horizon

$l^2 = -1/\Lambda$(reciprocal of the cosmological constant) ($l >> G$)

It is convenient to choose the additive constants in L_0 and $\overline{L}_0$, so that they vanish for the $M = J = 0$ black hole. One then has

$$M = \frac{1}{l}(L_0 + \overline{L}_0) \tag{23}$$

while the angular momentum is

$$J = L_0 - \overline{L}_0 . \tag{24}$$

This is notthe same as AdS_3 (three–dimensional anti-de Sitter space)metric which has negative mass $M = -\frac{1}{8G}$.

One wishes to count the number of excitations of the AdS_3 (three–dimensional anti-de Sitter space) vacuum with mass M and angular momentum J in the semiclassical regime of large M. The explicit computation of the central charge $c$is

$$c = \frac{3l}{2G} \tag{25}$$

According to (23) and (25) large M implies

$$n_R + n_L >> c, \tag{26}$$

where n_R (n_L) is the eigenvalue of L_0 ($\overline{L}_0$). The asymptotic growth of the number states of a conformal field theory with central charge c is then given by (J. A. Cardy, Nucl 1986).

$$S = 2\pi\sqrt{\frac{cn_R}{6}} + 2\pi\sqrt{\frac{cn_L}{6}}\ . \tag{27}$$

Using (25) , (23) and (24) , the expression for blackhole microstates can be written as

$$S = \pi\sqrt{\frac{l(lM+J)}{2G}} + \pi\sqrt{\frac{l(lM-J)}{2G}}, \tag{28}$$

And it is in agreement with the Bekenstein-Hawkingblack hole entropy result (22) for the BTZ (Banados-Teitelboim- Zanelli) black holeentropy.

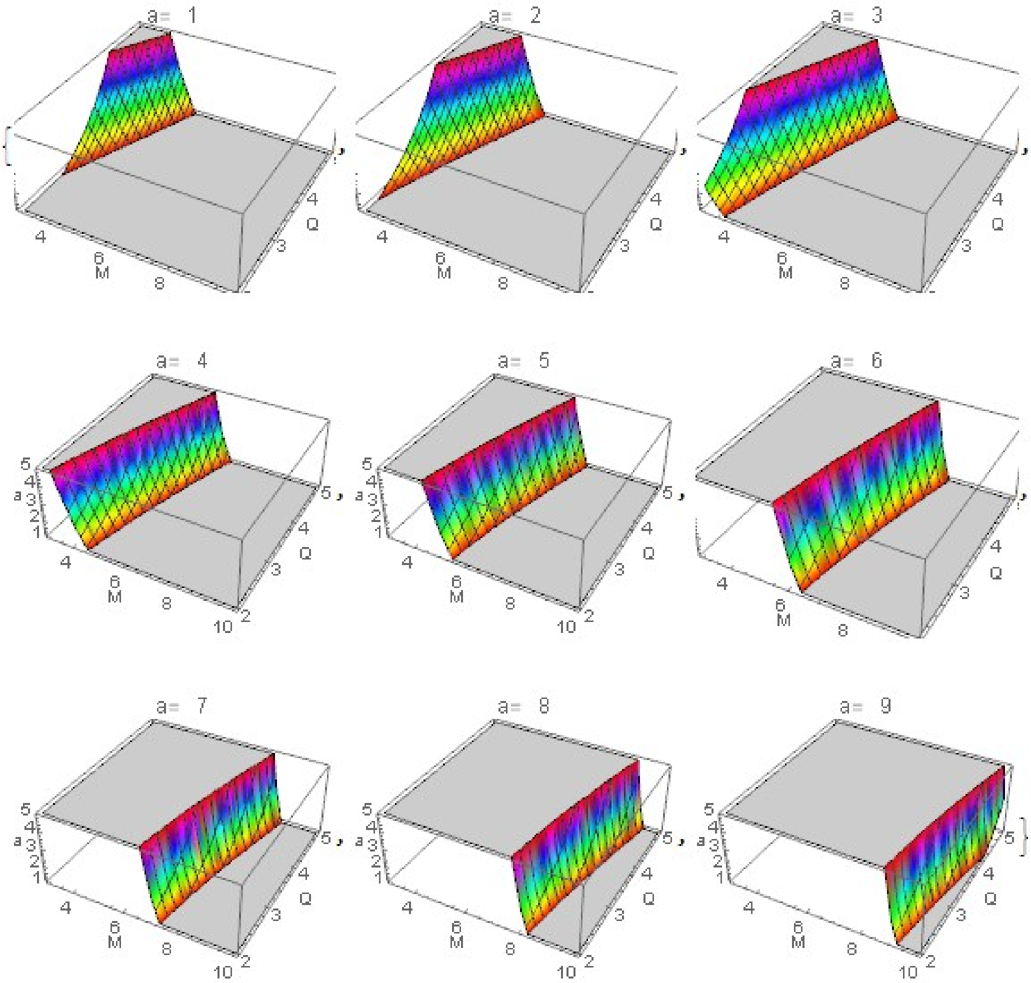

Figure 1 Snapshot Evolution of the surface gravity of the Kerr-Newmann black hole with the constraint $M^2>Q^2>a^2$.

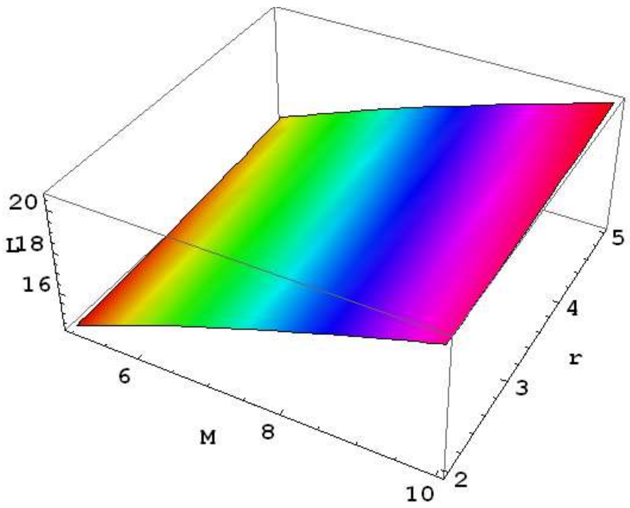

Figure 2 The Profile of Black Hole Entropy in terms of Radius of Event Horizon,M and *l*.

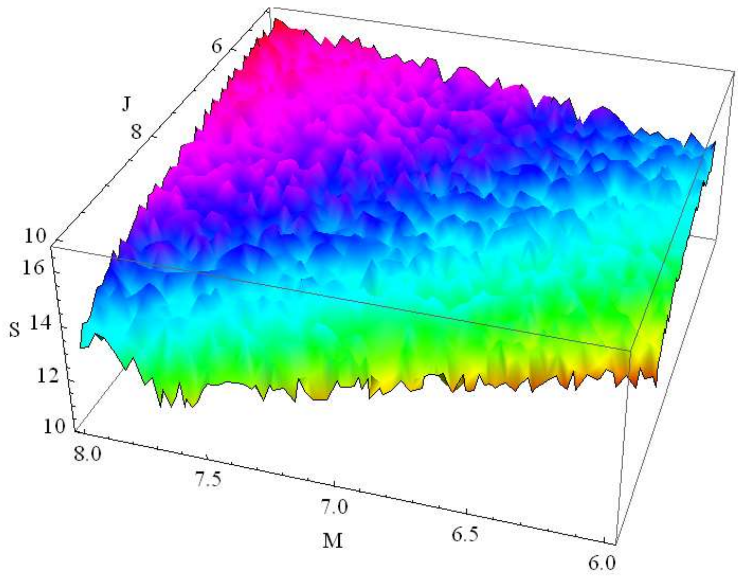

Figure 3 3D Profile of Black hole entropy in terms of M and J.

Concluding Remarks

In this present works, the detailed derivation for black hole entropy in classical aspect has been given and it is found to be the one fourth of the black hole's surface area. Quantum mechanical derivation and some interesting physical interpretations are to be implemented.

Theoretical validity of S=F(A), i.e., the entropy is simply a function of area has been proved using the basic thermodynamic entities like temperature, mass, surface gravity, area of a black hole.. The nature of surface gravity has been visualized in snapshot 3-D evolution of the surface gravity of the Kerr-Newmann black hole with the constraint $M^2 > Q^2 > a^2$ and the discrete nature of the evolution of surface gravity can be observed.The blackhole microstates treatment has been worked out and the result is in well agreement with BTZ blackhole result.

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